1. Asymptotics (Cont'd)

Today we cover some properties of asymptotics.

An easy property:

If $f = O(g)$, then $g = \Omega(f)$.

<u>Proof</u>: By $f = O(g)$, $\exists\, c, n_0 > 0$, s.t.

$\forall n > n_0,\ f(n) \leq c \cdot g(n)$.

$\Rightarrow \forall n > n_0,\ g(n) \geq \frac{1}{c} \cdot f(n)$

i.e. for $c' = \frac{1}{c}$. $n_0 = n_0$, $\forall n > n_0$,

$g(n) \geq c' \cdot f(n)$.

This is the def. for $g = \Omega(f)$

#

- $f = O(g) \Leftrightarrow g = \Omega(f)$
- $f = o(g) \Leftrightarrow g = \omega(f)$
- $f = \Theta(g) \Leftrightarrow f = O(g) \wedge g = O(f)$

  $\Leftrightarrow f = O(g) \wedge f = \Omega(g)$

Good ex. to try proving!

Transitivity:

$$a \le b,\ b \le c \Rightarrow a \le c$$

$$f = O(g),\ g = O(h) \Rightarrow f = O(h)\,?$$

Proof: $f = O(g) \Rightarrow \exists\, c_1, n_1 > 0$ s.t. $\forall n > n_1,\ 0 \le f(n) \le c_1 \cdot g(n)\}$

$g = O(h) \Rightarrow \exists\, c_2, n_2 > 0$, s.t. $\forall n > n_2,\ 0 \le g(n) \le c_2 \cdot h(n)\}$

Take $n_3 = \max(n_1, n_2)$, $c_3 = c_1 \cdot c_2$

Then: $\forall n \ge n_3,\ 0 \le f(n) \le c_1 g(n) \le c_1 \cdot c_2 h(n) = c_3 h(n)$ #

NOT ALL RULES APPLY!

"O" $\approx$ "$\le$" "not O" $\approx$ "not $\le$" $\not\approx$ "$>$"

$5 \not\le 3 \Rightarrow 5 > 3$ ✓

$\sin x \not\le 0 \Rightarrow \sin x > 0$ ✗

"not always $\le$" $\neq$ "always $>$"

e.g. $f(n) = n$ $g(n) = \begin{cases} n^2 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$

$f(n) = O(g(n))$? ✗

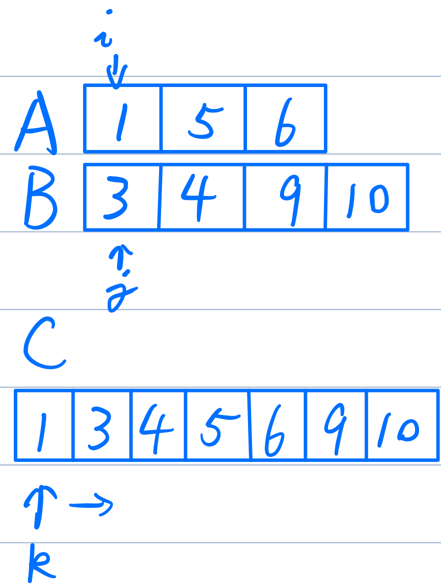
$g(n) = O(f(n))$? ✗

2. Merge Sort

Divide & Conquer (Gauss 1805?)

```
MergeSort(A[1,...,n]):
    MergeSort(A[1,...,n/2])
    MergeSort(A[n/2+1,...,n])
    Merge A[1,...,n/2] & A[n/2+1,...,n]
```

```
Merge(A, m, B, n, C) // A[1,...,m] & B[1,...,n] sorted
    i, j = 1
    For k = 1 to (m+n)
        If A[i] ≤ B[j]
            C[k] = A[i]
            i = i+1
        Else
            C[k] = B[j]
            j = j+1
        EndIf
    EndFor
```



Proof of Correctness for Merge:

Loop Invariant:

At the top of the For loop, $C[1,\ldots,k-1]$ contains the $k-1$ smallest elements of A & B in sorted order, and these elements are $A[1,\ldots,i-1]$, $B[1,\ldots,j-1]$.

- Initialization: $i=j=k=1$, holds trivially.
- Maintenance: Assume inv. holds at top of iteration k, w.l.o.g. say $A[i] \le B[j]$, then $A[i]$ is the smallest element not yet copied to C. Why?

$$A[i] \le A[i+1] \le \cdots \le A[m] \;\&$$
$$A[i] \le B[j] \le \cdots \le B[n]$$

✓ k smallest elements

✓ sorted order

✓ comes from $A[1,\ldots,i-1]$ and $B[1,\ldots,j-1]$

- Termination: $k=m+n+1$

$C[1,\ldots,m+n]$ contains the $\underline{(m+n)}$ all of them! elements of A & B in sorted order.

$\boxtimes$